Computer Science 320 Practice Final Exam
NAME: ________________________________

All questions to be answered on this exam. Please indicate if you are using the back of a page for your answer.
90 marks

Terminology:
Bal = the language over $\{), (\}$ of balanced strings of parentheses.
$A^nB^n = \{a^nb^n : n \geq 0\}$
$\text{Prime}_\Sigma = \{w \in \Sigma^* : |w| \text{ is prime }\}$
$H = \{< M, w >$: Turing Machine M halts on string $w\}$.
M_H is of a Turing Machine that semi-decides H. If M_H exists, its encoding is $< M_H >$.
IndepSet(G, k) is true iff G has a set of k **or fewer** vertices such that no edge in G has both ends in the set.
CLIQUE(G, k) is true iff G has a set of k **or more** pairwise-adjacent vertices (that is, a set $V' \subseteq V$ where $|V'| \geq k$ and for all $u \in V'$ and $v \in V'$ where $u \neq v$, (u, v) is an edge in G.)
$A_{TM} = \{< M, w > \mid M$ is a Turing Machine which accepts the string $w\}$.
$E_{TM} = \{< M > \mid M$ is a Turing Machine which accepts no strings$\}$.
$EQ_{TM} = \{< M_1, M_2 > \mid M_1$ and M_2 are Turing Machines that accept the same language$\}$.
The Pumping Lemma: If L is a regular language then there is a pumping constant p for L such that, for all strings w where $|w| \geq p$, w can be divided into three parts, $w = xyz$, where: (a) $|y| \geq 1$, (b) $|xy| \leq p$, and (c) for all $i \geq 0$, $xy^iz \in L$.

1. (9 marks) Answer True or False, by placing a T or F beside the statement. Answer on this sheet.

   ___ There is a deterministic TM that decides membership in the empty language

   ___ There is a deterministic TM that decides membership in $\{< M >$: M is a TM that accepts the empty language $\}$

   ___ There is a deterministic TM that decides membership in $NoBigClique = \{< G, k >$: G is a graph, and k is an integer, and G has **no** clique of size k **or more** $\}$.

   ___ There is a polynomial time deterministic TM that decides membership in $SmallClique = \{< G, k >$: G is a graph, and k is a positive integer, and G has a clique of size k **or less** $\}$.

   ___ There are an uncountably infinite number of regular languages

   ___ If $L_1 \cup L_2$ is context-free then it must be the case that at least one of L_1 and L_2 are context-free.

   ___ If a PDA were modified so it had a queue in place of its stack, the result would be Turing-equivalent to a Turing Machine.

   ___ There are a countably infinite number of languages over any given (finite) alphabet.

2. (5 marks) Give a PDA for the language $L = \{w \in \{a,b\}^* :$ if the middle symbol of w is a, then w must be a palindrome, i.e, the latter half of w is the reverse of the first half $\}$. Note that this language is the **union** of $\{waw^r : w \in \{a,b\}^*\}$ and $\{w : w$ has odd length and the middle symbol in w is $b\}$.

3. (5 marks) Give a Context-Free Grammar for the Kleene-star closure of the language in the previous question.

4. (5 marks) Let $\bigoplus$ denote "exclusive or" for languages: that is, $L_1 \bigoplus L_2 = \{w : w \in L_1$ or $w \in L_2$ but not both $\}$. Prove **constructively** that the class of regular languages are closed under $\bigoplus$ by doing the following: given that L_1 and L_2 are regular languages with FAs M_1 and M_2, respectively, show that $L \bigoplus L_2$ is also regular by giving a construction for the FA for $L_1 \bigoplus L_2$.

5. (4 marks) Find the equivalent deterministic FA for the DFA below, using the method given in class. Include all useful states, and no states that are not useful (i.e., not reachable from the start state).

   $M = (\{q_1, q_2, q_3, q_4\}, \delta, q_1, \{q_4\})$, where the transition funciton δ is given by the table below:

state	symbol	destination state
q_1	a	q_2
q_1	b	q_4
q_2	ϵ	q_3
q_2	a	q_3
q_3	b	q_4
q_4	ϵ	q_1
q_4	a	q_1

6. (8 marks) Prove that the language $\{w \in \{a,b\}^* :$ the number of a's in w is an integer multiple of the number of b's $\}$ is not regular. Zero is an integer multiple of all integers.

7. (5 marks) Give a CFG for the language $\{a^i b^j c^{i+j+k} d^k : i, j, k \geq 0\}$.

8. (5 marks) Give a PDA for the same language as in the previous question.

9. Recall: "integers" includes both positive and negative integers, and zero.

   (a) (3 marks) Show that the integers are countably infinite.

   (b) (3 marks) Show that the ordered pairs of integers (not necessarily positive) are countably infinite.

10. (5 marks) Consider the following language: **StateUse** $= \{< M_1, q, w > |M$ is a Turing Machine which contains state q, and and M enters state q when run on input $w.\}$

    Is StateUse decidable? Prove your answer.

11. (10 marks) Suppose you are working on the GlassHalfEmpty problem. You have shown that it is in NP, and you want to see whether it there is a polynomial time algorithm to solve it. You can only make reductions to or from the problems GlassHalfEmpty, CLIQUE, HamiltonPath, and GraphConnectivity, where GraphConnectivity is the problem of determining whether an input undirected graph G is connected, that is, for every pair of vertices a, b in the graph, there is a path between a and b. Recall that "P1 reduces to P2 by a polynomial-time mapping" is written "P1 $\leq_P$ P2".

    Below are all the candidate relations **(not all are true)**. Beside each relation,
    ...write **TRUE** if the statement is known to be true
    ...write **FALSE** if the statement is known to be false
    ...write **P=NP** if the statement implies that P=NP
    ...write **P $\neq$ NP** if the statement implies that P$\neq$NP
    ...write $GHE \in$ **P** if the statement implies that GlassHalfEmpty is in P
    ...write $GHE \notin$ **P** if the statement implies that GlassHalfEmpty is not in P
    ...write $GHE \in$ **NP-c** if the statement implies that GlassHalfEmpty is NP-complete
    ...write $GHE \notin$ **NP-c** if the statement implies that GlassHalfEmpty is not NP-complete

    CLIQUE$\leq_P$ HamiltonPath ________________________________

    CLIQUE$\leq_P$ GraphConnectivity________________________________

    CLIQUE$\leq_P$ GlassHalfEmpty ________________________________

    GlassHalfEmpty$\leq_P$ CLIQUE________________________________

    GlassHalfEmpty$\leq_P$ GraphConnectivity________________________________

    GlassHalfEmpty $\leq_P$ HamiltonPath ________________________________

    GraphConnectivity$\leq_P$ CLIQUE________________________________

    GraphConnectivity$\leq_P$ HamiltonPath ________________________________

    GraphConnectivity$\leq_P$ GlassHalfEmpty ________________________________
    HamiltonPath$\leq_P$ CLIQUE________________________________

    HamiltonPath$\leq_P$ GraphConnectivity________________________________

    HamiltonPath$\leq_P$ GlassHalfEmpty ________________________________

12. Suppose L is a language for which there exists a Printer-TM.

    (a) (3 marks) What can we conclude about L: is it polynomially decidable, decidable, semidecidable, or undecidable? Breifly explain why.

    (b) (5 marks) Suppose further that there is a Turing computable function that, for every string w, returns a number r such "if w is in L then the Printer-TM prints it out within the first r strings." What can we now conclude about L: is it polynomially decidable, decidable, semidecidable, or undecidable? Breifly explain why.

13. (5 marks) In this question, we look at two ways of constraining the problem SET PARTITION. Recall that SET PARTITION is the following problem:

    Problem SET PARTITION:
    Given: a multiset of integers $S = \{x_1, x_2, \ldots, x_n\}$
    Decide: Is there a way to partition S so that both partitions sum to the same amount?

    We showed this problem was NP-c, by reducing SUBSET SUM to it.

    NP-completeness results are made even stronger by adding constraints to the problem. For example, it has been proved that: "SUBSET SUM remains NP-c even if all the set elements are positive integers. I.e., POSITIVE SUBSET SUM is NP-complete." Certain constraints, however, will render the problem polynomial.

    (a) (2 marks) Prove that ALL EQUAL SET PARTITION is polynomial, where the problem is:

        Problem ALL EQUAL SET PARTITION:
        Given: a multiset of integers $S = \{x_1, x_2, \ldots, x_n\}$ where all x_i are equal
        Decide: Is there a way to partition S so that both partitions sum to the same amount?

    (b) (5 marks) Prove that POSITIVE SET PARTITION is NP-c, where the problem is:
        Problem POSITIVE SET PARTITION:
        Given: a multiset of integers $S = \{x_1, x_2, \ldots, x_n\}$ where all x_is are positive
        Decide: Is there a way to partition S so that both partitions sum to the same amount?
        For full marks, you should include indications as to why the reduction works (that is, returns the correct answer) and why you believe it is poly-time.

14. **Problem:** HAMPATH
    Given: An undirected graph G with vertex set V and edge set E, where $|V| = n$ and $|E| = m$.
    Decide: Is there a path H in G that visits each vertex exactly once? (Such a path corresponds to a permutation $v_1, v_2, \ldots, v_n$ of the vertices such that $(v_i, v_{i+1}) \in E$ for all i where $1 \leq i < n$.)
    Problem: HAMCYC
    Given: An undirected graph G with vertex set V and edge set E, where $|V| = n$ and $|E| = m$.
    Decide: Is there a cycle C in G that visits each vertex exactly once? (Such a cycle corresponds to a permutation $v_1, v_2, \ldots, v_n$ of the vertices such that $(v_i, v_{i+1}) \in E$ for all i where $1 \leq i < n$; and such that $(v_n, v_1) \in E$.)

(a) (4 marks) Prove that HAMPATH $\leq_P$ HAMCYC. Briefly explain why the reduction is **correct**, and why it is polynomial time.

(b) (4 marks) Prove that HAMCYC $\leq_P$ HAMPATH. Briefly explain why the reduction is **correct**, and why it is polynomial time.