

Computer Science 320 Practice Final Exam  
NAME: \_\_\_\_\_

All questions to be answered on this exam. Please indicate if you are using the back of a page for your answer.  
90 marks

**Terminology:**

Bal = the language over  $\{\}, \{\}$  of balanced strings of parentheses.

$A^n B^n = \{a^n b^n : n \geq 0\}$

$\text{Prime}_\Sigma = \{w \in \Sigma^* : |w| \text{ is prime} \}$

$H = \{ \langle M, w \rangle : \text{Turing Machine } M \text{ halts on string } w \}$ .

$M_H$  is of a Turing Machine that semi-decides  $H$ . If  $M_H$  exists, its encoding is  $\langle M_H \rangle$ .

$\text{IndepSet}(G, k)$  is true iff  $G$  has a set of  $k$  **or fewer** vertices such that no edge in  $G$  has both ends in the set.

$\text{CLIQUE}(G, k)$  is true iff  $G$  has a set of  $k$  **or more** pairwise-adjacent vertices (that is, a set  $V' \subseteq V$  where  $|V'| \geq k$  and for all  $u \in V'$  and  $v \in V'$  where  $u \neq v$ ,  $(u, v)$  is an edge in  $G$ .)

$A_{TM} = \{ \langle M, w \rangle \mid M \text{ is a Turing Machine which accepts the string } w \}$ .

$E_{TM} = \{ \langle M \rangle \mid M \text{ is a Turing Machine which accepts no strings} \}$ .

$EQ_{TM} = \{ \langle M_1, M_2 \rangle \mid M_1 \text{ and } M_2 \text{ are Turing Machines that accept the same language} \}$ .

The Pumping Lemma: If  $L$  is a regular language then there is a pumping constant  $p$  for  $L$  such that, for all strings  $w$  where  $|w| \geq p$ ,  $w$  can be divided into three parts,  $w = xyz$ , where:  
(a)  $|y| \geq 1$ , (b)  $|xy| \leq p$ , and (c) for all  $i \geq 0$ ,  $xy^i z \in L$ .

1. (9 marks) Answer True or False, by placing a T or F beside the statement. Answer on this sheet.

\_\_\_ There is a deterministic TM that decides membership in the empty language

\_\_\_ There is a deterministic TM that decides membership in  $\{ \langle M \rangle : M \text{ is a TM that accepts the empty language} \}$

\_\_\_ There is a deterministic TM that decides membership in  $NoBigClique = \{ \langle G, k \rangle : G \text{ is a graph, and } k \text{ is an integer, and } G \text{ has no clique of size } k \text{ or more} \}$ .

\_\_\_ There is a polynomial time deterministic TM that decides membership in  $SmallClique = \{ \langle G, k \rangle : G \text{ is a graph, and } k \text{ is a positive integer, and } G \text{ has a clique of size } k \text{ or less} \}$ .

\_\_\_ There are an uncountably infinite number of regular languages

\_\_\_ If  $L_1 \cup L_2$  is context-free then it must be the case that at least one of  $L_1$  and  $L_2$  are context-free.

\_\_\_ If a PDA were modified so it had a queue in place of its stack, the result would be Turing-equivalent to a Turing Machine.

\_\_\_ There are a countably infinite number of languages over any given (finite) alphabet.

2. (5 marks) Give a PDA for the language  $L = \{w \in \{a, b\}^* : |w| \text{ is odd, and if the middle symbol of } w \text{ is } a, \text{ then } w \text{ must be a palindrome, i.e., the latter half of } w \text{ is the reverse of the first half}\}$ .  
Note that this language is the **union** of  $\{waw^r : w \in \{a, b\}^*\}$  and  $\{w : w \text{ has odd length and the middle symbol in } w \text{ is } b\}$ .
3. (5 marks) Give a Context-Free Grammar for the Kleene-star closure of the language in the previous question.
4. (5 marks) Let  $\oplus$  denote “exclusive or” for languages: that is,  $L_1 \oplus L_2 = \{w : w \in L_1 \text{ or } w \in L_2 \text{ but not both}\}$ . Prove **constructively** that the class of regular languages are closed under  $\oplus$  by doing the following: given that  $L_1$  and  $L_2$  are regular languages with FAs  $M_1$  and  $M_2$ , respectively, show that  $L_1 \oplus L_2$  is also regular by giving a construction for the FA for  $L_1 \oplus L_2$ .
5. (4 marks) Find the equivalent deterministic FA for the NFA below, using the method given in class. Include all useful states, and no states that are not useful (i.e., not reachable from the start state).

$M = (\{q_1, q_2, q_3, q_4\}, \delta, q_1, \{q_4\})$ , where the transition function  $\delta$  is given by the table

	state	symbol	destination state
	$q_1$	$a$	$q_2$
	$q_1$	$b$	$q_4$
below:	$q_2$	$\epsilon$	$q_3$
	$q_2$	$a$	$q_3$
	$q_3$	$b$	$q_4$
	$q_4$	$\epsilon$	$q_1$
	$q_4$	$a$	$q_1$

6. (8 marks) Prove that the language  $\{w \in \{a, b\}^* : \text{the number of } a\text{'s in } w \text{ is an integer multiple of the number of } b\text{'s}\}$  is not regular. Zero is an integer multiple of all integers.
7. (5 marks) Give a CFG for the language  $\{a^i b^j c^{i+j+k} d^k : i, j, k \geq 0\}$ .
8. (5 marks) Give a PDA for the same language as in the previous question.
9. Recall: “integers” includes both positive and negative integers, and zero.
  - (a) (3 marks) Show that the integers are countably infinite.
  - (b) (3 marks) Show that the ordered pairs of integers (not necessarily positive) are countably infinite.
10. (5 marks) Consider the following language: **StateUse** =  $\{ \langle M_1, q, w \rangle \mid M \text{ is a Turing Machine which contains state } q, \text{ and } M \text{ enters state } q \text{ when run on input } w. \}$

Is StateUse decidable? Prove your answer.

11. (10 marks) Suppose you are working on the GlassHalfEmpty problem. You have shown that it is in NP, and you want to see whether there is a polynomial time algorithm to solve it. You can only make reductions to or from the problems GlassHalfEmpty, CLIQUE, HamiltonPath, and GraphConnectivity, where GraphConnectivity is the problem of determining whether an input undirected graph  $G$  is connected, that is, for every pair of vertices  $a, b$  in the graph, there is a path between  $a$  and  $b$ . Recall that “P1 reduces to P2 by a polynomial-time mapping” is written “ $P1 \leq_P P2$ ”.

Below are all the candidate relations (**not all are true**). Beside each relation,

...write **TRUE** if the statement is known to be true

...write **FALSE** if the statement is known to be false

...write **P=NP** if the statement implies that  $P=NP$

...write  **$P \neq NP$**  if the statement implies that  $P \neq NP$

...write  $GHE \in P$  if the statement implies that GlassHalfEmpty is in  $P$

...write  $GHE \notin P$  if the statement implies that GlassHalfEmpty is not in  $P$

...write  $GHE \in NP\text{-c}$  if the statement implies that GlassHalfEmpty is NP-complete

...write  $GHE \notin NP\text{-c}$  if the statement implies that GlassHalfEmpty is not NP-complete

CLIQUE  $\leq_P$  HamiltonPath \_\_\_\_\_

CLIQUE  $\leq_P$  GraphConnectivity \_\_\_\_\_

CLIQUE  $\leq_P$  GlassHalfEmpty \_\_\_\_\_

GlassHalfEmpty  $\leq_P$  CLIQUE \_\_\_\_\_

GlassHalfEmpty  $\leq_P$  GraphConnectivity \_\_\_\_\_

GlassHalfEmpty  $\leq_P$  HamiltonPath \_\_\_\_\_

GraphConnectivity  $\leq_P$  CLIQUE \_\_\_\_\_

GraphConnectivity  $\leq_P$  HamiltonPath \_\_\_\_\_

GraphConnectivity  $\leq_P$  GlassHalfEmpty \_\_\_\_\_

HamiltonPath  $\leq_P$  CLIQUE \_\_\_\_\_

HamiltonPath  $\leq_P$  GraphConnectivity \_\_\_\_\_

HamiltonPath  $\leq_P$  GlassHalfEmpty \_\_\_\_\_

12. Suppose  $L$  is a language for which there exists a Printer-TM.

- (3 marks) What can we conclude about  $L$ : is it polynomially decidable, decidable, semidecidable, or undecidable? Briefly explain why.
- (5 marks) Suppose further that there is a Turing computable function that, for every

string  $w$ , returns a number  $r$  such “if  $w$  is in  $L$  then the Printer-TM prints it out within the first  $r$  strings.” What can we now conclude about  $L$ : is it polynomially decidable, decidable, semidecidable, or undecidable? Briefly explain why.

13. (5 marks) In this question, we look at two ways of constraining the problem SET PARTITION. Recall that SET PARTITION is the following problem:

Problem SET PARTITION:

Given: a multiset of integers  $S = \{x_1, x_2, \dots, x_n\}$

Decide: Is there a way to partition  $S$  so that both partitions sum to the same amount?

We showed this problem was NP-c, by reducing SUBSET SUM to it.

NP-completeness results are made even stronger by adding constraints to the problem. For example, it has been proved that: “SUBSET SUM remains NP-c even if all the set elements are positive integers. I.e., POSITIVE SUBSET SUM is NP-complete.” Certain constraints, however, will render the problem polynomial.

- (a) (2 marks) Prove that ALL EQUAL SET PARTITION is polynomial, where the problem is:

Problem ALL EQUAL SET PARTITION:

Given: a multiset of integers  $S = \{x_1, x_2, \dots, x_n\}$  where all  $x_i$  are equal

Decide: Is there a way to partition  $S$  so that both partitions sum to the same amount?

- (b) (5 marks) Prove that POSITIVE SET PARTITION is NP-c, where the problem is:

Problem POSITIVE SET PARTITION:

Given: a multiset of integers  $S = \{x_1, x_2, \dots, x_n\}$  where all  $x_i$ s are positive

Decide: Is there a way to partition  $S$  so that both partitions sum to the same amount?

For full marks, you should include indications as to why the reduction works (that is, returns the correct answer) and why you believe it is poly-time.

14. **Problem:** HAMPATH

**Given:** An undirected graph  $G$  with vertex set  $V$  and edge set  $E$ , where  $|V| = n$  and  $|E| = m$ .

**Decide:** Is there a path  $H$  in  $G$  that visits each vertex exactly once? (Such a path corresponds to a permutation  $v_1, v_2, \dots, v_n$  of the vertices such that  $(v_i, v_{i+1}) \in E$  for all  $i$  where  $1 \leq i < n$ .)

**Problem:** HAMCYC

**Given:** An undirected graph  $G$  with vertex set  $V$  and edge set  $E$ , where  $|V| = n$  and  $|E| = m$ .

**Decide:** Is there a cycle  $C$  in  $G$  that visits each vertex exactly once? (Such a cycle corresponds to a permutation  $v_1, v_2, \dots, v_n$  of the vertices such that  $(v_i, v_{i+1}) \in E$  for all  $i$  where  $1 \leq i < n$ ; and such that  $(v_n, v_1) \in E$ .)

- (a) (4 marks) Prove that  $\text{HAMPATH} \leq_P \text{HAMCYC}$ . Briefly explain why the reduction is **correct**, and why it is polynomial time.
- (b) (4 marks) Prove that  $\text{HAMCYC} \leq_P \text{HAMPATH}$ . Briefly explain why the reduction is **correct**, and why it is polynomial time.