Digital Logic and Computer Organization

Boolean Algebra

Logic Motivation Question

- There are two villages, A and B. People in village A always tell the truth, while people in village B always tell lies. A traveller standing on the fork leading to village A and B and wants to go to village A (of course). But which way to go? Two people standing there, one from village A and the other from village B. They know each other, but the traveller doesn't. The traveller is allowed to ask one question only in order to find the right path to village A. Which question should the traveller ask?

Boolean Motivation Question

- Only one person from one of the villages stands at the fork road.
- The person can only answer true or false.
- The person either tells truth or lies about all facts.
- The person follows all the logic operation rules.
- Form a boolean statement and determine which path leading to the truth village according to the T/F answer assessed by the person.

Boolean Algebra

- George Boole (1854)
  - introduced a systematic treatment of logic
- Huntington (1904)
  - defined Boolean algebra by providing 6 postulates (basic assumptions of a mathematical system) that must be satisfied:
    (1) Closure
    (2) Identity
    (3) Commutativity
    (4) Distributivity
    (5) Complement
    (6) Distinct Elements (|U| >= 2)
- Shannon (1938)
  - applied Boolean algebra to relay circuitry found in telephone routing switches, with Distinct Elements (|U| = 2)

Switching Algebra

- Two valued Boolean Algebra
  - defined on a set with two elements and two binary operators + (or) and • (and) which satisfy Huntington's Postulates
- Set of elements: B = {0, 1}
- 0 and 1 are the complements of each other: 0' = 1; 1' = 0
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x	y	x•y	x+y
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	1

Switch Algebra

- Postulate 1 (Closure): result of an operation is in the set
- Postulate 2 (Identity): $x+0 = x$; $x \bullet 1 = x$
- Postulate 3 (Commutativity): $x+y = y+x$; $x \bullet y = y \bullet x$
- Postulate 4 (Distributivity): $x+y \bullet z = (x+y) \bullet (x+z)$;
  $x \bullet (y+z) = x \bullet y+x \bullet z$
- Postulate 5 (Complement): $x+x' = 1$; $x \bullet x' = 0$
- Postulate 6 (Distinct elements): $0 \neq 1$

Proof (Expression Equivalence)

- Perfect Induction: construct truth tables (literal analysis)
- Axiomatic Proof: apply Huntington's Postulates and its derived theorems
- Duality Principle
  - the dual of an expression is found by replacing all instances of "+" with "•", all instances of "•" with "+", all instances of "0" with "1", and all instances of "1" with "0"
  - if A = B then Dual(A) = Dual(B)
- Proof by Contradiction

Theorems

- Idempotency: $x+x = x$; $x \bullet x = x$
- Annulment: $x+1 = 1$; $x \bullet 0 = 0$;
- Absorption: $x+x \bullet y = x$; $x \bullet (x+y) = x$
- Involution: $(x')' = x$
- DeMorgan's: $(x \bullet y)' = x'+y'$; $(x+y)' = x' \bullet y'$
- Adjacency: $x \bullet y + x \bullet y' = x$; $(x+y) \bullet (x+y') = x$